

Novel Sets of Coupling Expansion Parameters for low-energy pQCD¹

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Abstract

In quantum theory, physical amplitudes are usually presented in the form of Feynman perturbation series in powers of coupling constant α . However, it is known that these amplitudes are not regular functions at $\alpha = 0$.

For QCD, we propose new sets of expansion parameters $\mathbf{w}_k(\alpha_s)$ that reflect singularity at $\alpha_s = 0$ and should be used instead of powers α_s^k . Their explicit form is motivated by the so called Analytic Perturbation Theory. These parameters reveal saturation in a strong coupling case at the level $\alpha_s^{eff}(\alpha_s \gg 1) = \mathbf{w}_1(\alpha_s \gg 1) \sim 0.5$. They can be used for quantitative analysis of diverse low-energy amplitudes.

We argue that this new picture with non-power sets of perturbation expansion parameters, as well as the saturation feature, is of a rather general nature.

1 Subject and Motivation

In quantum theory, physical amplitudes are usually presented in the form of perturbative series in powers of an expansion parameter g related to intensity of interaction, the non-linear term in the equation of motion. As it is known from the early 50s [2], these amplitudes are not regular functions at $\alpha = 0$, irrelevant to the existence of UV divergencies and renormalization. The most general and transparent argument [3] can be formulated via a representation of path integral.

In QFT, one deals with Feynman perturbation theory (PT) series in powers of a numerical parameter α . In particular, in current practice, such a series for a QCD observable serves as a launch pad for Renorm-Group (RG) invariant expansion in powers of invariant/effective coupling $\bar{\alpha}_s(Q^2)$ or $\bar{\alpha}_s(s)$. As a rule, this function $\bar{\alpha}_s$, being a sum of ultraviolet (UV) logs, obeys phantom singularities, like the so called Landau pole. In QCD, being located at a scale of a few hundred MeV, they spoil the low-energy applications.

From the mathematical point of view, a possibility of convergent power expansion implies that the expanded function $f(\alpha)$ is a regular (analytic) function of its argument at small α . Meanwhile, it is known for sure [2] (also Refs.[4, 5]) that in the complex α plane, there is an essential singularity at the origin $\alpha = 0$. Correspondingly, at a small real positive α , the PT series $\sum_n c_n \alpha^n$ is divergent with $c_n \sim n!$ at $n \gg 1$. Nevertheless, under some condition, a finite sum $\sum_n^N c_n \alpha^n$ can serve as a means for numerical approximation of the expanded function $f(\alpha)$.

¹A preliminary version with the main results was published in Ref.[1].

In such a situation, the value $n^* \sim 1/\alpha$ is a critical one. Here, PT expansion can start to explode. Indeed, in the low-energy QCD, at $\alpha_s \sim 0.2 - 0.3$, $n^* \sim 3 - 5$ and thus the value of the 3- and 4-loop calculation is under question. Examples are known (see, *e.g.* Table 2 in refs.[6] and [7]).

How serious is this menace for practical low-energy pQCD calculation? Is it possible to use some other expansion parameter $\mathbf{w}(\alpha)$ instead of α or a non-power set $\{\mathbf{w}_k(\alpha)\}$ of parameters² instead of $\{\alpha^k\}$?

Below, we try to answer this question by using a combination of rather general arguments including the principles of causality and renormalizability as well as self-consistency condition of theoretical description with respect to conversion from one physical picture (representation) to another by a suitable integral transformation.

During the last decade, on the basis of these principles, a special scheme for ghost-free calculations in QCD was proposed [9] and elaborated[10]. It is known now as Analytic Perturbation Theory (APT). For fresh reviews, we recommend Refs.[7, 11]. In what follows, we shall use the APT notation and results. Compendium of a few relevant APT definitions is presented below in Appendix A.

Discussing, in Conclusion, the possible meaning of our particular results, we involve additional evidence from the soluble QFT models with an infrared fixed point.

2 Equivalence of Transformations

Within the Analytic Perturbation Theory, a transition from the common QCD effective coupling function $\alpha_s(L)$, $L = \ln(Q^2/\Lambda^2)$ to the Euclidean $\alpha_E(L)$ or Minkowskian $\alpha_M(L)$ one *etc.* can be treated as a transition to a new expansion parameter. For example, in the 1-loop case, at $\alpha_s, L > 0$ the conversion

$$\alpha_s \rightarrow \alpha_M(\alpha_s) = \mathbf{w}^M(\alpha) = \frac{\arctan(\pi\beta_0\alpha_s)}{\pi\beta_0} \sim \alpha_s - \frac{\pi^2\beta_0^2}{3}\alpha_s^3 + \dots$$

induces a transition to the new effective coupling

$$\alpha_s(L) = \frac{1}{\beta_0 L} \rightarrow \alpha_M(L) = \frac{\arctan(\pi/L)}{\pi\beta_0} \sim \alpha_s(L) - \frac{\pi^2\beta_0^2}{3}[\alpha_s(L)]^3 + \dots$$

Generally, all the APT non-power expansion functions, Minkowskian $\mathfrak{A}_k(s)$, Euclidean $\mathcal{A}_k(Q^2)$ or Distance $\mathfrak{N}_k(r^{-2})$, are mapped via the relation

$$L \rightarrow \Phi(\alpha_s) = - \int^{\alpha_s} \frac{da}{\beta(a)} = \frac{1}{\beta_0\alpha_s} + \frac{\beta_1}{\beta_0^2} \ln\left(\frac{\beta_1}{\beta_0} + \frac{1}{\alpha_s}\right) + O(\beta_2) \quad (1)$$

$L = \ln(s/\Lambda^2), \ln(Q^2/\Lambda^2), \ln(1/r^2\Lambda^2)$, on sets $\{\mathbf{w}_k^{\text{APT=M,E,D}}(\alpha_s)\}$

$$\mathfrak{A}_k \rightarrow \mathbf{w}_k^M(\alpha_s), \quad \mathcal{A}_k \rightarrow \mathbf{w}_k^E(\alpha_s), \quad \mathfrak{N}_k \rightarrow \mathbf{w}_k^D(\alpha_s).$$

²As it was proposed, *e.g.*, by Caprini and Fischer [8].

The functions \mathbf{w}_k^{APT} resultant from different sources are related by integral transformations that stem from ones connecting the “parent” APT expansion functions. For example, from Adler and Fourier transformations

$$\mathcal{A}_k(Q^2) = Q^2 \int_0^\infty \frac{ds \mathfrak{A}_k(s)}{(s + Q^2)^2}, \quad \mathfrak{N}_k(r) = r \int_0^\infty dQ \sin(Qr) \mathcal{A}_k(Q^2)$$

and eq.(1) there follows a general relation

$$\mathbf{w}_k^E(\alpha) = \int_0^\infty da K_{EZ}(\alpha, a) \mathbf{w}_k^Z(a); \quad Z = M, D. \quad (2)$$

2.1 The one-loop case

In the one-loop case, consequent elements are connected by simple recurrent relation

$$\mathbf{w}_{k+1}^{(1)}(\alpha) = \frac{\alpha^2}{k} \frac{d \mathbf{w}_k^{(1)}(\alpha)}{d \alpha}, \quad (3)$$

while eq.(2) for $Z = M$ takes the form

$$\mathbf{w}_k^E(\alpha) = \frac{1}{2} \int_{-\infty}^\infty \frac{da}{a^2} \frac{\mathbf{w}_k^M(a)}{1 + \cosh(1/\beta_0 \alpha - 1/\beta_0 a)}. \quad (4)$$

The novel “Minkowskian” and “Euclidean” one-loop expansion functions

$$\alpha_M^{(1)}(\alpha) = \mathbf{w}_1^{M,(1)} = \frac{\arctan(\pi \beta_0 \alpha)}{\pi \beta_0}; \quad \mathbf{w}_1^{E,(1)}(\alpha) = \alpha + \frac{\beta_0^{-1}}{1 - e^{1/(\beta_0 \alpha)}}; \quad (5)$$

$$\mathbf{w}_2^{M,(1)} = \frac{\alpha^2}{1 + \pi^2 \beta_0^2 \alpha^2}; \quad \mathbf{w}_2^{E,(1)}(\alpha) = \alpha^2 - \frac{\beta_0^{-2} e^{1/(\beta_0 \alpha)}}{[1 - e^{1/(\beta_0 \alpha)}]^2}; \quad (6)$$

$$\mathbf{w}_3^{M,(1)}(\alpha) = \frac{\alpha^3}{[1 + (\pi \beta_0 \alpha)^2]^2}; \quad \mathbf{w}_4^{M,(1)}(\alpha) = \frac{\alpha^4 (3 - \pi^2 \beta_0^2 \alpha^2)}{3 [1 + (\pi \beta_0 \alpha)^2]^3}; \quad (7)$$

are mutually connected by relations (3), (4).

All the functions $\mathbf{w}_k^{\dots}(\alpha_s)$ as $\alpha_s \rightarrow \infty$ have finite limits

$$\mathbf{w}_1^M(\infty) = \alpha_M(\infty) = \mathbf{w}_1^E(\infty) = \alpha_E(\infty) = \frac{1}{2\beta_0} \sim 0.7; \quad \mathbf{w}_2^{M,(1)}(\infty) = \frac{1}{\pi^2 \beta_0^2} \sim 0.20;$$

$$\mathbf{w}_2^{E,(1)}(\infty) = \frac{1}{12\beta_0^2} \sim 0.16; \quad \mathbf{w}_3^{M,(1)}(\infty) = 0; \quad \mathbf{w}_4^{M,(1)}(\alpha) = -\frac{1}{(\pi \beta_0)^4} \sim -0.04.$$

Here, the limit $\alpha_s \rightarrow \infty$, by (1), is adequate to $L \rightarrow +0$. Curious enough, a physical region below Λ (*i.e.*, $L < 0$) corresponds to negative α_s values.

2.2 Some properties of functions $\mathbf{w}_k^{\text{APT}}(\alpha_s)$

A few general properties of novel expansion functions are of interest

- Like their APT “parents” $\mathfrak{A}_k, \mathcal{A}_k, \aleph_k$, functions $\{\mathbf{w}_k^{\text{APT}}(\alpha_s)\}$ *in the whole real positive domain* $(-\infty < \alpha_s < +\infty)$ form non-power sets of oscillating functions with k zeroes.
- Natural scales for them are $\alpha_M^* = \frac{1}{\pi\beta_0} \sim 0.5$, $\alpha_E^* = \frac{1}{\beta_0} \sim 1.4$.
- Some of the functions, like \mathbf{w}_k^E , obey singularity $e^{1/(\beta_0 \alpha_s)}$.
- They are not sensitive to “their family origin”. In Fig.1, curves $\mathbf{w}_{1,2}^E$ are close to $\mathbf{w}_{1,2}^M$ (as compared to the Caprini-Fischer curves $w_{1,2}^{cf}$ obtained[8] by conformal transformation of the Borel image). As can be shown, the same is true for $\mathbf{w}_{1,2}^D$.

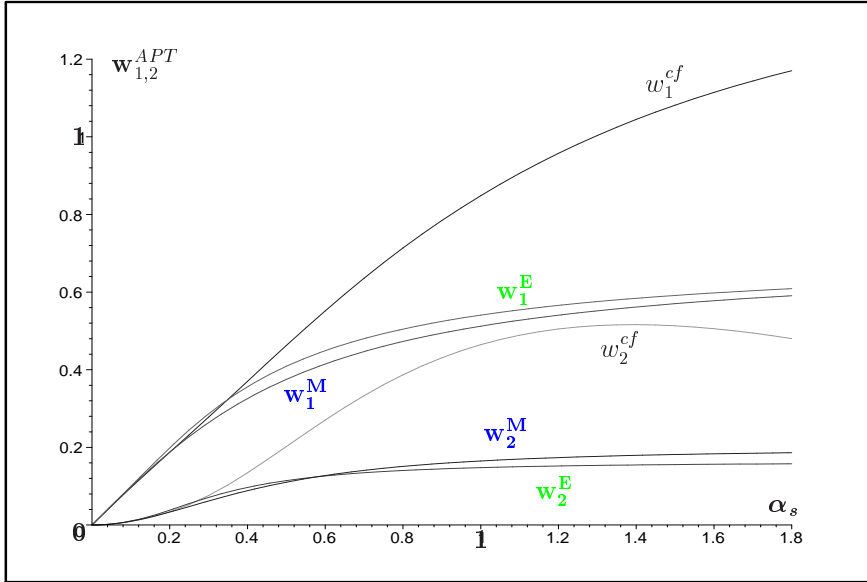


Figure 1: *Comparison of the first two APT-inspired $\mathbf{w}_{1,2}^{E,M}(\alpha_s)$ functions with the Caprini-Fischer $w_{1,2}^{cf}$ ones.*

Some qualitative properties of $\mathbf{w}_k^{\text{APT}}(\alpha_s)$ should be mentioned

1. APT-inspired functions $\mathbf{w}_{1,2}^{\text{APT}}(\alpha_s)$ deviate from powers α_s, α_s^2 at $\alpha_s \sim 0.3 - 0.4$, which corresponds to a few GeV region.

2. Quick saturation of the first $\mathbf{w}_1(\alpha_s)$ at 0.4 and second $\mathbf{w}_2(\alpha_s)$ at ~ 0.15 values.

Thus, “strong coupling” means $\alpha_{eff} \lesssim 0.5$. Physically, due to this, in a few GeV region effective QCD coupling should be less than 0.5.

3. Relative difference between functions “of various origin” is small (less than 10 per cent) up to $\alpha_s \sim 0.8$. Due to this, as a first step, for crude quantitative estimate one could use one-loop Minkowskian expressions (5), (6), (7).

4. Note also that a modification of the PT expansion by new prescription $(\alpha_s)^k \rightarrow \mathbf{w}_k(\alpha_s)$ for 1-argument observable, like total cross-sections or Adler functions, leads – by use of the RG algorithm – to a non-power APT result.

3 Discussion

- First, the *possible use of novel functions* $\{\mathbf{w}_k^{\text{APT}}(\alpha_s)\}$ in pQCD has to be mentioned. In practice, the RG improving of PT results is limited by the “1-argument” objects, like total cross-sections and D-functions. For the “2-argument” ones (diffraction amplitudes, structure functions), one is forced to use special tricks, *e.g.*, projection on 1-argument moments.

New RG-inspired expansions over $(\alpha_s)^k \rightarrow \mathbf{w}_k^{\text{APT}}(\alpha_s)$ provide another bypass solution to this issue. In other words, we recommend using novel expansion functions for theoretical analysis of divers physical amplitudes in the low-energy (low momentum transfer) regions. For a semi-quantitative quick analysis one could use one-loop Minkowskian functions (A5), (A6) with effective values [13] of the Λ parameter.

- We believe that the observed feature of *interaction saturation* could have a rather *general nature*. Indeed, the saturation of the interaction intensity or, rather, its self-saturation in the “strong coupling limit” could be correlated with analogous features of some soluble QFT models, like massless two-dimensional[14, 15] Thirring and sine-Gordon model[16] equivalent to the massive Thirring one[17]. Additional evidence can be gathered from models with infrared fixed point, like the Gross-Neveu model [18, 19] and the 3-dimensional φ^4 model [20].
- One of the possible ways of further analysis, to reveal this aspired generality, could be connected with RG study of the corresponding non-quantized field models by the recently devised[21, 22]) *method of renormgroup symmetries*(=RGS) for boundary value problems of classical mathematical physics. Here, one has to find appropriate RGS invariants and then relate them with a quantized version with the help of the functional integral representation and the saddle-point procedure [23].

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Appendix A Basic APT formulae

Within the Analytic Perturbation Theory, the set of common QCD coupling functions and its powers is changed for a nonpower set of ghost-free functions connected by recurrent relations. For instance, in the one-loop case, instead of the polynomial set

$$\bar{\alpha}_s^{(1)}(Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}; \quad (\bar{\alpha}_s^{(1)}(Q^2))^2 = \frac{1}{\beta_0^2 \ln^2 L}; \quad (\bar{\alpha}_s^{(1)}(Q^2))^3 \quad \dots \quad (\text{A1})$$

one deals with

$$\alpha_E^{(1)}(Q^2) \equiv \mathcal{A}_1(Q^2) = \frac{1}{\beta_0 L} + \frac{\Lambda^2}{\beta_0 (\Lambda^2 - Q^2)}; \quad \mathcal{A}_2 = \frac{1}{\beta_0^2 L^2} - \frac{\Lambda^2 Q^2}{\beta_0^2 (\Lambda^2 - Q^2)^2}; \quad \mathcal{A}_3; \quad \dots \quad (\text{A2})$$

related by the differential relation

$$-\frac{1}{k} \frac{\mathcal{A}_k^{(1)}(Q^2)}{dL} = \beta_0 \mathcal{A}_{k+1}^{(1)}(Q^2); \quad L = \ln \frac{Q^2}{\Lambda^2}. \quad (\text{A3})$$

The functions $\{\mathcal{A}_k(Q^2)\}$ form a basis for expansion of RG-invariant functions depending on one kinematic argument, $Q^2 = \mathbf{Q}^2 - Q_0^2$, the transferred momentum squared. For example, the Adler function is presented there in the form of non-polynomial perturbation expansion

$$D(Q^2) = \sum_k d_k \mathcal{A}_k(Q^2).$$

These ‘‘Euclidean’’ expansion functions are related to the common $[\bar{\alpha}_s]^k$ ones by the prescription

$$\mathcal{A}_k(Q^2) = \int_0^\infty \frac{\rho_k(\sigma)}{\sigma + Q^2} d\sigma, \quad \rho_k(\sigma) = \frac{1}{\pi} \text{Im}[\bar{\alpha}_s(-\sigma - i\varepsilon)]^k, \quad (\text{A4})$$

that provides correspondence in the weak coupling limit: $\mathcal{A}_k \rightarrow \alpha_s^k$ as $\alpha_s \rightarrow 0$.

At the same time, within the APT, one can define ghost-free ‘‘Minkowskian’’ expansion functions for RG-invariant observables in another representation, for observable depending on s , c.m. energy squared, like (relation of) total cross-section(s)

$$R(s) = \sum_k d_k \mathfrak{A}_k(s); \quad \mathfrak{A}_k(s) = \int_s^\infty \frac{d\sigma}{\sigma} \rho_k(\sigma).$$

These Minkowskian functions are connected with the Euclidean ones by integral transformations

$$\mathfrak{A}_k(s) = \frac{i}{2\pi} \int_{s-i\varepsilon}^{s+i\varepsilon} \frac{dz}{z} \mathcal{A}_k(-z); \quad \mathcal{A}_k(Q^2) = Q^2 \int_0^\infty \frac{\mathfrak{A}_k(s) ds}{(s + Q^2)^2}.$$

The first of them, Minkowskian effective coupling, in the one-loop case has a simple form

$$\alpha_M^{(1)}(s) = \mathfrak{A}_1^{(1)}(s) = \frac{1}{\pi\beta_0} \arccos \frac{L_s}{\sqrt{L_s^2 + \pi^2}} \Big|_{L_s > 0} = \frac{1}{\pi\beta_0} \arctan \frac{\pi}{L_s}, \quad L_s = \ln \frac{s}{\Lambda^2}. \quad (\text{A5})$$

Accordingly,

$$\mathfrak{A}_2^{(1)}(s) = \frac{1}{\beta_0^2} \frac{1}{L^2 + \pi^2}; \quad \mathfrak{A}_3^{(1)}(s) = \frac{1}{\beta_0^3} \frac{L}{(L^2 + \pi^2)^2}; \dots \quad (\text{A6})$$

Quite analogously, one can devise[24, 25] analogous expansion functions \aleph_k for the “distance picture”³

$$\aleph_k\left(\frac{1}{r^2}\right) = r \int_0^\infty dQ \sin(Qr) \mathcal{A}_k(Q) = \frac{2}{\pi} \int_0^\infty \frac{dQ}{Q} \sin(Qr) \int_0^\infty \frac{d\sigma \rho(\sigma)}{\sigma + Q^2}. \quad (\text{A7})$$

The convenient form of the APT formalism uses a *spectral density* $\rho(\sigma)$ taken from perturbative input (A4). Then all the involved functions in the mentioned pictures look like

$$\mathcal{A}_k(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho_k(\sigma) d\sigma}{\sigma + Q^2}, \quad \mathfrak{A}_k(s) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \rho_k(\sigma), \quad \aleph_k\left(\frac{1}{r^2}\right) = \int_0^\infty \frac{\rho_k(\sigma) d\sigma}{\sigma} \left(1 - e^{-r\sqrt{\sigma}}\right). \quad (\text{A8})$$

In the 1-loop case

$$\rho_1^{(1)} = \frac{1}{\beta_0 [L_\sigma^2 + \pi^2]}; \quad k \beta_0 \rho_{k+1}^{(1)}(\sigma) = -\frac{d\rho_k^{(1)}(\sigma)}{dL_\sigma}; \quad L_\sigma = \ln \frac{\sigma}{\Lambda^2}$$

As it was noted above, these expressions were generalized for the higher-loop case with transitions across heavy quark thresholds and successively used (see, e.g., Ref.[6]) for fitting of various data.

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³On the base of 3-dimensional Fourier transformation

$$\bar{\psi}(Q) = (2\pi)^{-2} \int d\mathbf{r} \psi(r) e^{i\mathbf{Q}\mathbf{r}}.$$

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